5.9.1. Quantified Basic Sentences: Truth Tree Problems

A. For each of the following arguments, build a **truth tree** to show that the argument is valid.

1. $Ga \;\; \therefore \; \exists x\, Gx$
2. $(Ga \rightarrow \sim Ha) \;\; \therefore \;\; (\forall x\, Gx \rightarrow \; \sim \forall x\, Hx)$
3. $\forall x\, Gx \;\; . \;\; (Ga \rightarrow Ha) \;\; \therefore \; \exists x\, Hx$
4. $(\forall x\, Gx \;\vee\; \sim \exists x\, Gx) \;\; . \;\; Ga \;\; \therefore \; \forall x\, Gx$
5. $(Ga \rightarrow \forall x\, Gx) \;\; . \;\; (Ha \rightarrow \exists x \sim Gx) \;\; \therefore \; \sim(Ga \wedge Ha)$
6. $(\exists x\, Gx \;\wedge\; \exists x \sim Gx) \;\; \therefore \;\; \sim(\forall x\, Gx \;\vee\; \forall x \sim Gx)$

B. For each of the following arguments, build a **truth tree** to decide whether or not the argument is **valid**. If the argument is invalid, provide a model that's a validity counterexample for that argument.

1. $(\forall x\, Gx \rightarrow \; \forall x\, Hx) \;\; . \;\; Ga \;\; \therefore \;\; Ha$ [i]

2. $(\exists x\, Gx \rightarrow \; \forall x\, Hx) \;\; . \;\; Ga \;\; \therefore \;\; Ha$ [v]

3. $(\forall x\, Gx \rightarrow \; \forall x\, Hx) \;\; \therefore \;\; (\exists x\, Gx \; \rightarrow \; \exists x\, Hx)$ [i]
If bowling's always legal then mahjongg's always legal. ∴ If bowling's ever legal then mahjongg's sometimes legal.

4. $(\exists x\, Gx \; \rightarrow \; \exists x\, Hx) \;\; \therefore \;\; (\forall x\, Gx \rightarrow \; \forall x\, Hx)$ [i]
If bowling's ever legal then mahjongg's sometimes legal. ∴ If bowling's always legal then mahjongg's always legal.

5. $\forall x\, Hx \;\; . \;\; (Ga \rightarrow \; \exists x \sim Hx) \;\; \therefore \;\; \sim Ga$ [v]
[Everybody's happy. If Kitty's singing, then somebody's not happy. ∴ Kitty's not singing.]

6. Jack's a surfer but Neko's not. ∴ Someone's a surfer but not everyone is.

7. If anyone can solve the problem, Lucretia can. (But) Lucretia can't solve the problem. $\therefore$ No one can solve the problem.

8. If anything's physical then everything is. $\therefore$ Either everything's physical or nothing is.

9. If everyone can solve the problem, Lucretia can. (But) Lucretia can't solve the problem. $\therefore$ No one can solve the problem.

10. $(\forall x\ Gx \land \forall x\ Hx)$ $\therefore$ $(\exists x\ Gx \land \exists x\ Hx)$ [v]

11. $((\exists x\ Gx \lor \exists x\ Hx) \rightarrow \forall x\ Ix)$. $(Ga \lor \sim\exists x\ Jx)$. Jb
$\therefore$ $(\forall x\ Gx \rightarrow \forall x\ Hx)$ [v]

C. For each of the following sentences, build a **truth tree** to show that it's **logically true**.

1. $(\forall x\ Gx \rightarrow Ga)$

2. $(Ga \rightarrow \exists x\ Gx)$

3. $(\forall x\ Gx \lor \exists x \sim Gx)$

4. $(\exists x\ Gx \lor \exists x \sim Gx)$

5. $\sim(\forall x\ Gx \land \exists x \sim Gx)$

6. $\sim(\forall x\ Gx \land \forall x \sim Gx)$

7. $((\exists x\ Gx \land \exists x \sim Gx) \rightarrow (\sim\forall x\ Gx \land \sim\forall x \sim Gx))$

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1	$(\exists x\ Gx \land \exists x \sim Gx)$	**1**	$\exists x\ (Gx \land \exists x \sim Gx)$
0	$\therefore$ $\exists x\ (Gx \land \exists x \sim Gx)$	**0**	$\therefore$ $(\exists x\ Gx \land \exists x \sim Gx)$

For each of the numbered sentences below, state which are instances of the following universal sentence.

$$\forall \mathbf{x}\ ((Gx \wedge Hx) \rightarrow \exists x\ {\sim}Gx)$$

1. $((Ga \wedge Ha) \rightarrow \exists x\ {\sim}Gx)$
2. $((Ga \wedge Ha) \rightarrow \exists x\ {\sim}Ga)$
3. $\forall x\ ((Gx \wedge Hx) \rightarrow {\sim}Ga)$
4. $((Gb \wedge Hb) \rightarrow \exists x\ {\sim}Gx)$
5. $((Ga \wedge Hb) \rightarrow \exists x\ {\sim}Gx)$
6. $((Ga \wedge Hx) \rightarrow \exists x\ {\sim}Gx)$

B. For each of the numbered sentences below, state which have "$\mathbf{(Ga \wedge Hb)}$" as an instance.

1. $\exists x\ (Ga \wedge Hx)$
2. $\exists x\ (Gx \wedge Hb)$
3. $\exists x\ (Gx \wedge Hx)$
4. $(\exists x\ Ga \wedge Hb)$
5. $\exists x\ (Ga \wedge Hb)$

C. Based on your answer to **(A)**, state whether the universal sentence "$\forall \mathbf{x}\ ((Gx \wedge Hx) \rightarrow \exists x\ {\sim}Gx)$"is true or false in the following model.

a: Neko **G**__: is a cat
b: Rex **H**__: is fat

$\mathbb{D}$: {**Neko**, **Rex**}

a: **Neko** G: {**Neko**}
b: **Rex** H: {**Neko, Rex**}

D. For each of the existential sentences picked in your answer to **(B)**, state whether that sentence is true or false in the above model.

E. According to our definition of "instance," is "Ga" an instance of "$\exists x\ Ga$"?

F. For each of the quantified sentences below, state whether it is true or false in the model given here.

$\mathbb{D}$: {**2**, **3**, **4**}

a: **2**	**G**: {**4**}
b: **3**	**H**: {**3**, **4**}
c: **4**	**I**: {**2**, **3**, **4**}
	J: { }

(1) $\exists x\, Hx$
(2) $\forall x\, Hx$
(3) $\exists x\, Jx$
(4) $\exists x\,(Hx \wedge Gx)$
(5) $\exists x\,(Hx \vee Jx)$
(6) $\exists x\,(Hx \wedge \sim Hx)$
(7) $\exists x\,(Hx \vee \sim Hx)$
(8) $\forall x\,(Hx \vee \sim Hx)$
(9) $\forall x\,(Jx \vee \sim Jx)$
(10) $(Ga \rightarrow \exists x\, Gx)$
(11) $(Gb \rightarrow \exists x\, Gx)$
(12) $\forall x\,(Ix \rightarrow Hx)$
(13) $\forall x\,(Hx \rightarrow Ix)$
(14) $\exists x\,(Gx \rightarrow Jx)$
(15) $\exists x\,(Jx \rightarrow Gx)$
(16) $\exists x\,(Jx \leftrightarrow Gx)$
(17) $\forall x\,(Jx \leftrightarrow Gx)$
(18) $\forall x\,(Jx \rightarrow Gx)$
(19) $\forall x\,(Jx \rightarrow \sim Gx)$
(20) $\exists x\,((Hx \vee Jx) \leftrightarrow Gx)$
(21) $\forall x\,((Hx \vee Jx) \leftrightarrow Gx)$
(22) $\forall x\,((Hx \vee Ix) \leftrightarrow \sim Jx)$
(23) $\forall x\,((Hx \vee Ix) \leftrightarrow Ix)$
(24) $\forall x\,((Hx \wedge Ix) \leftrightarrow Ix)$
(25) $\exists x\,((Hx \wedge Ix) \wedge \sim Gx)$
(26) $\forall x\,((Hx \vee \sim Hx) \leftrightarrow \sim Jx)$
(27) $\forall x\,((Hx \vee \sim Hx) \leftrightarrow Ix)$
(28) $\forall x\,((Hx \vee \sim Hx) \leftrightarrow Gx)$

G. We noted that the sentence "**∃x (Gx ∧ ∃x ~Gx)**" would, intuitively, be true wherever there were at least two objects, one G and one non-G. And in fact we can establish semantically that this sentence is logically equivalent to "**(∃x Gx ∧ ∃x ~Gx)**" by showing that each sentence entails the other.

Consider what a validity counterexample for each argument would look like.

1	(∃x Gx ∧ ∃x ~Gx)	**1**	∃x (Gx ∧ ∃x ~Gx)
0	∴ ∃x (Gx ∧ ∃x ~Gx)	**0**	∴ (∃x Gx ∧ ∃x ~Gx)

For the argument on the left it's especially easy to see the problem: to make the premise "(∃x Gx ∧ ∃x ~Gx)" true a model will need one object in the extension of "G" (to make "∃x Gx" true) and a second object not in the extension of "G" (to make "∃x ~Gx" true).

$\mathbb{D}$: {2, 3}

a: 2 **G**: {2}
b: 3

But the conclusion "∃x (Gx ∧ ∃x ~Gx)" has two instances in such a model.

1 (Ga ∧ ∃x ~Gx)
0 (Gb ∧ ∃x ~Gx)

Since "(Ga ∧ ∃x ~Gx)" is true here, the model makes the conclusion "∃x (Gx ∧ ∃x ~Gx)" true, and so isn't a validity counterexample. And no modification of the model will change this: **(i)** simple replacing a with b in the extension of "G" will make the second instance "(Gb ∧ ∃x ~Gx)" true, leaving the conclusion with a true instance. Putting both objects either **(ii)** in the extension of "G" or **(iii)** outside the extension of "G" will make either "∃x ~Gx" or "∃x Gx" false, and so make the premise false. And **(iv)** leaving the objects as they are but adding more objects leaves the premise and conclusion true.

Provide a similar **semantic explanation for why the right argument** must be **valid**.

H. Return once more to the model where Neko is a cat and Rex isn't one.

G__: is a cat

$\mathbb{D}$: {**Neko**, **Rex**}

a: **Neko** **G**: {**Neko**}
b: **Rex**

The discussion of instances noted that if our account of "instance" involved replacing **every** occurrence of "x" in the scope formula (whether free or not), then we wrongly count the consistent existential sentence "**∃x (Gx ∧ ∃x ~Gx)**" a contradiction.

But suppose a critic replies that we should instead count as instances **both** the sentences following the 'only free variables' condition **and** those ignoring that condition. By that more relaxed standard "**∃x (Gx ∧ ∃x ~Gx)**" would have four instances in this model.

(i) **(Ga ∧ ∃x ~Ga)** (iii) **(Gb ∧ ∃x ~Ga)**
(ii) **(Ga ∧ ∃x ~Gb)** (iv) **(Gb ∧ ∃x ~Gb)**

Since "**∃x (Gx ∧ ~Gx)**" still has at least one true instance in this model – Sentences (ii) and (iii) – it is rightly not counted as a contradiction on this account of instances.

Show that this more lax standard for being an instance leads to incorrect results, using as test case the following universal sentence.

∀x (Gx → ∃x ~Gx)

G__: __is a cat

For every object: if that object is a cat, then there's some object which isn't a cat.

On our account of instances, the scope formula "(Gx → ∃x ~Gx)" has two instances in our model (repeated here).

G__: is a cat

𝔻: {**Neko**, **Rex**}

a: **Neko** **G**: {**Neko**}
b: **Rex**

Instances of "∀x (Gx → ∃x ~Gx)" in this model:

(1) (Ga → ∃x ~Gx)
(2) (Gb → ∃x ~Gx)

To see that both of these conditionals are true in this model, it suffices to note that the consequent "∃x ~Gx" is true in this model – for there is indeed an object (Rex) which isn't a cat. But conditional semantics dictates that the whole **conditional is true whenever its consequent is true**.

●	▲	(● → ▲)
1	**1**	**1**
1	0	0
0	**1**	**1**
0	0	1

That makes sense intuitively: in a situation where at least one object is non-G, it will be true of any object we pick that if it's G, something *isn't G*.

But according to the more lax alternative account of instances, the sentence "∀x (Gx → ∃x ~Gx)" will have four instances in this model.

☠ **Instances of "∀x (Gx → ∃x ~Gx)" in this model?** ☠

(1) (Ga → ∃x ~Gx) (3) (Ga → ∃x ~Ga)
(2) (Gb → ∃x ~Gx) (4) (Gb → ∃x ~Gb)

Will all these be true in this model?